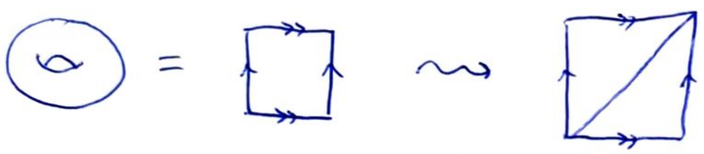
Math 132: Differential Topology

§ Euler characteristics and triangulations

A <u>triangulation</u> of a manifold is a subdivision into simplices, which are higher dimensional generalization of triangles.

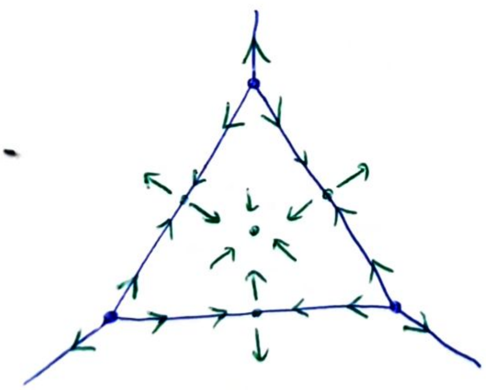
<u>Ex</u> A triangulation of T^2



<u>Thm</u> If M is an m-dim'l triangulated manifold, then

$$\chi(M) = \sum_{j=0}^{m} (-1)^j \cdot \#(j\text{-dim'l faces})$$

proof) Given a triangulation, we can construct a vector field v on M with exactly one zero for each face, with index $(-1)^j$:



The theorem then follows from Poincaré-Hopf. ∎

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§ Cobordisms (extra topic)

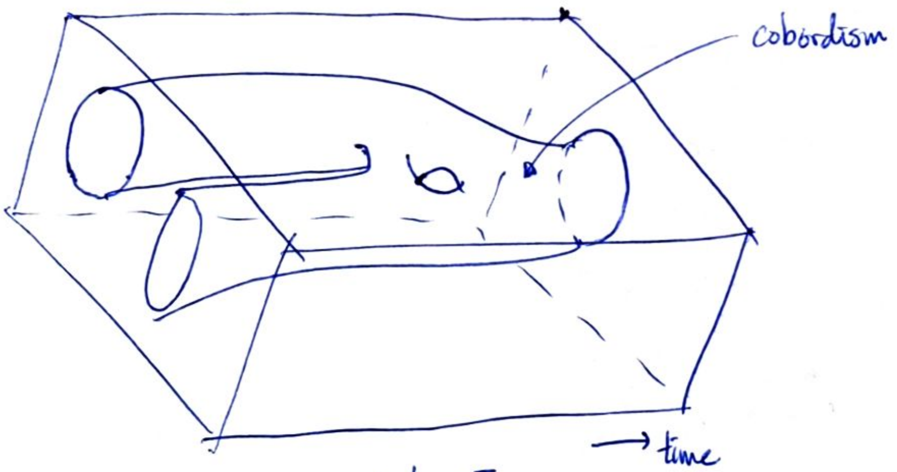
Much of our discussions so far can be generalized to intersection theory even when the dimensions are not complementary.

Suppose $M_1, M_2 \subset N$ are compact manifolds (of not necessarily complementary dim).

Then, after some homotopy to make the intersection transverse,

$M_1 \cap M_2$ is a manifold of dimension $m_1 + m_2 - n$.

Under homotopy of M_1 (or M_2), $M_1 \cap M_2$ may change shape, but it remains in the same <u>cobordism class</u>:



$N \times I$

<u>Def</u> Two manifolds M, M' of the same dimension are <u>cobordant</u> if there is a compact manifold W with boundary $\partial W \cong M \sqcup M'$.

Two manifolds being cobordant is an equivalence relation, and we call the equivalence classes <u>cobordism classes</u>.

They form an abelian group under disjoint union.

Let Ω^{O}_n be the <u>unoriented cobordism group</u> of n-manifolds,
and Ω^{SO}_n be the <u>oriented cobordism group</u> of oriented n-manifolds (and oriented cobordisms).

The Cartesian product defines a multiplication, so

$$\Omega^{O}_{*} := \bigoplus_{n \geq 0} \Omega^{O}_n \quad \text{and} \quad \Omega^{SO}_{*} := \bigoplus_{n \geq 0} \Omega^{SO}_n$$

are graded algebras.

<u>Ex</u> $\underset{\text{generated by } \cdot}{\Omega^{O}_0 \cong \mathbb{Z}/2}$ and $\underset{\text{generated by } \overset{\cdot}{+}}{\Omega^{SO}_0 \cong \mathbb{Z}}$

These are where our mod-2 and oriented intersection theory took values in!

This is out of the scope for this course, but René Thom proved in 1954 that

<u>Thm</u> $\Omega^{O}_{*} \cong \mathbb{F}_2[x_i \mid i \geq 1,\ i \neq 2^j-1]$

polynomial algebra with one generator x_i in each dim $i \neq 2^j-1$

Explicitly, in low dimensions,

$\Omega^{O}_{0} \cong \mathbb{Z}/2$,
$\Omega^{O}_{1} \cong 0$,
$\Omega^{O}_{2} \cong \mathbb{Z}/2$,
$\Omega^{O}_{3} \cong 0$,
$\Omega^{O}_{4} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
$\Omega^{O}_{5} \cong \mathbb{Z}/2$, ...

for even i, we may take $x_i = [\mathbb{RP}^i]$

For the oriented cobordisms, it's known that $\Omega^{SO}_{*} \otimes \mathbb{Q} \cong \mathbb{Q}[y_{4i} \mid i \geq 1]$ (mod torsion)
where $y_{4i} = [\mathbb{CP}^{2i}]$

and $\Omega^{SO}_{0} \cong \mathbb{Z}$,
$\Omega^{SO}_{1} \cong 0$,
$\Omega^{SO}_{2} \cong 0$,
$\Omega^{SO}_{3} \cong 0$,
$\Omega^{SO}_{4} \cong \mathbb{Z}$,
$\Omega^{SO}_{5} \cong \mathbb{Z}/2$, ...